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Differential simplicity and the module of derivations

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Abstract

Let A be a Noetherian local ring containing a field k. If $\operatorname{Der}_k(A)$ is finite and A is differentially simple under a set of k-derivations then it is shown that $\operatorname{Der}_k(A)$ is free.

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Introduction

Let k be a field and let A be a Noetherian local ring containing k such that A is differentially simple simple under $\operatorname{Der}_k(A)$. If $\operatorname{Der}_k(A)$ is finitely generated as an A-module then we show that $\operatorname{Der}_k(A)$ is free as an A-module. In fact, we prove the following result which is more general.

Theorem 5. *Let A be a Noetherian local ring and let $\mathfrak{D}$ be an A-submodule of* $\operatorname{Der}(A)$. *Let I be the maximally $\mathfrak{D}$-differential ideal. If $\mathfrak{D}$ is closed under Lie operation of derivations and is finitely generated as an A-module then $\mathfrak{D}/I\mathfrak{D}$ is free as an A/I-module.*

This result has the following application: Assume that characteristic of k is zero. If A is a G-ring (see [5, p. 256] for definition) then, by [2], A is regular. It was shown by an example in [3] that A, in general, is not regular. Recently in [1], a much simpler example of this fact was given. We first show that in this example $\operatorname{Der}_k(A)$ is finitely generated. Now by our result $\operatorname{Der}_k(A)$ is free. This gives us an example of a Noetherian local ring A containing a field k of characteristic zero such that A is

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differentially simple under $\mathrm{Der}_k(A)$, $\mathrm{Der}_k(A)$ is finitely generated and free and A is not regular. This example is also of interest in view of Zariski Lipman conjecture (though it is not a counterexample as it does not belong to the right category).

We remark that if A is differentially simple then A contains a field k such that A is differentially simple under $\mathrm{Der}_k(A)$.

1. The results

By a ring we mean a commutative ring with unity.

Let A be a ring. We denote by $\mathrm{Der}(A)$, the module of derivations of A. More specifically, if k is a ring and A is a k-algebra then we denote by $\mathrm{Der}_k(A)$, the module of k-linear derivations of A.

Let $\mathfrak{D} \subset \mathrm{Der}(A)$. An ideal I of A is said to be $\mathfrak{D}$-differential if $d(I) \subset I$ for all $d \in \mathfrak{D}$. If ideals I and J are $\mathfrak{D}$-differential then so are the ideals $I+J, I \cap J$ and IJ.

An ideal I is said to be a maximally $\mathfrak{D}$-differential ideal if it is a maximal element of the set

$$\mathfrak{F}_{\mathfrak{D}} = \{J \mid J \text{ is a proper } \mathfrak{D}\text{-differential ideal of } A\}.$$

Note that by Zorn's lemma, $\mathfrak{F}_{\mathfrak{D}}$ contains maximal elements. Also note that if A is local then $\mathfrak{F}_{\mathfrak{D}}$ has a unique maximal element.

An ideal I is said to be maximally differential if it is maximally $\mathfrak{D}$-differential for some set $\mathfrak{D} \subset \mathrm{Der}(A)$.

A ring is said to be differentially simple if the ideal (0) is maximally differential.

If A is differentially simple under $\mathfrak{D} \subset \mathrm{Der}_k(A)$ then the set $\{a \in A \mid d(a) = 0 \text{ for all } d \in \mathfrak{D}\}$ is a field.

Definition 1 (cf. Maloo [4, 1.6]). Let $\mathfrak{D} \subset \mathrm{Der}(A)$ and let $a \in A$. Let $\mathfrak{D}_a$ denote the ideal of A generated by the set $\{d_1 \dots d_r(a) \mid r \geq 0; d_1, \dots, d_r \in \mathfrak{D}\}$. Then $a \in \mathfrak{D}_a$ and $\mathfrak{D}_a$ is $\mathfrak{D}$-differential.

We recall the following lemma from [4].

Lemma 2. *Let $\mathfrak{D} \subset \mathrm{Der}(A)$ and let I be a maximally $\mathfrak{D}$-differential ideal. For $a \in A \setminus I$, construct $\mathfrak{D}_a$ as in Definition 1. Then $I + \mathfrak{D}_a = A$.*

Definition 3 (cf. Maloo [4, 1.8]). Let $\mathfrak{D} \subset \mathrm{Der}(A)$ and let I be a maximally $\mathfrak{D}$-differential ideal. Let Q be a (prime) ideal of A containing I. To every $a \in A$ we associate a nonnegative integer $n := n(a, Q)$ as follows: If $a \in I$, put $n(a, Q) = \infty$. If $a \notin I$, construct $\mathfrak{D}_a$ as in Definition 1. Then by Lemma 2, $I + \mathfrak{D}_a = A$. Therefore, there exist $d_1, \dots, d_r \in \mathfrak{D}$ (for some $r \geq 0$) such that $d_1 \dots d_r(a) \notin Q$. Let $n(a, Q)$ denote the least nonnegative integer r with this property. Clearly, $n(a, Q)$ is uniquely determined.

Lemma 4. *Let $\mathfrak{D}$, I and Q be as in Definition* 3. *Then:*

(a) *For $a, b \in A$, $n(a+b, Q) \geq \min\{n(a,Q), n(b,Q)\}$ and equality holds if $n(a,Q) \neq n(b,Q)$. Also $n(ab,Q) \geq n(a,Q) + n(b,Q)$.*

(b) *Let $J = (a_1, \ldots, a_m)$ be an ideal of A. Then for all $a \in J$, $n(a,Q) \geq \min\{n(a_i,Q) \mid 1 \leq i \leq m\}$.*

Proof. Immediate from the definition. □

Theorem 5. *Let A be a Noetherian local ring and let $\mathfrak{D}$ be an A-submodule of* $\mathrm{Der}(A)$. *Let I be the maximally $\mathfrak{D}$-differential ideal. If $\mathfrak{D}$ is closed under Lie operation of derivations and is finitely generated as an A-module then $\mathfrak{D}/I\mathfrak{D}$ is free as an A/I-module.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of A. Let $d_1, \ldots, d_m$ be a minimal set of generators for $\mathfrak{D}$. We have to show that if $a_1, \ldots, a_m \in A$ such that $\sum_{i=1}^{m} a_i d_i = 0$ then $a_i \in I$ for all $i = 1, \ldots, m$. Assume the contrary. For $i = 1, \ldots, m$, let $n_i = n(a_i, \mathfrak{m})$, as defined in Definition 3. Let $n = \min\{n_1, \ldots, n_m\}$. Then $n < \infty$. Choose $a_1, \ldots, a_m$ such that n is the least. We show that $n = 0$. Suppose $n > 0$. We may assume that $n = n_1$. Then there exist $\delta_1, \ldots, \delta_n \in \mathfrak{D}$ such that $\delta_1 \ldots \delta_n(a_1) \notin \mathfrak{m}$. Put $\delta = \delta_n$. Then

$$
\begin{aligned}
0 &= \left[\delta, \sum_{i=1}^{m} a_i d_i\right] \\
&= \sum_{i=1}^{m} \{\delta(a_i) d_i + a_i[\delta, d_i]\} \\
&= \sum_{i=1}^{m} (\delta(a_i) + b_i) d_i,
\end{aligned}
$$

for some $b_i \in J = (a_1, \ldots, a_m)$, $i = 1, \ldots, m$. Then by Lemma 4 $n(\delta(a_1)+b_1, \mathfrak{m}) = n-1$ and $n(\delta(a_i) + b_i, \mathfrak{m}) \geq n - 1$ for $i = 2, \ldots, m$. This contradicts the minimality of n. Therefore $n = 0$. This is impossible as $a_i \in \mathfrak{m}$ and therefore $n_i \geq 1$ for all $i = 1, \ldots, m$. □

Corollary 6. *Let A and $\mathfrak{D}$ be as in Theorem* 5. *Suppose A is differentially simple under $\mathfrak{D}$. Then $\mathfrak{D}$ is free as an A-module.*

Proof. Immediate from Theorem 5. □

Corollary 7. *Let A be a Noetherian local ring containing a field k such that* $\mathrm{Der}_k(A)$ *is finitely generated as an A-module. If A is differentially simple under* $\mathrm{Der}_k(A)$ *then* $\mathrm{Der}_k(A)$ *is free as an A-module.*

Proof. Follows from Corollary 6 as $\mathrm{Der}_k(A)$ is closed under the Lie operation of derivations. □

Remark 8. We apply Corollary 7 to show that there exists a Noetherian local ring A containing a field k of characteristic zero such that $\mathrm{Der}_k(A)$ is finitely generated and free as an A-module and A is not regular. To show that we need the following lemma:

Lemma 9. *Let A be a Noetherian domain containing a field k and let K be the quotient field of A. If $\mathrm{Der}_k(K)$ is finitely generated as a K-vector space then $\mathrm{Der}_k(A)$ is finitely generated as an A-module.*

Proof. Let $n = \dim_K \mathrm{Der}_k(K)$. Let $d_1, \dots, d_n$ be a basis of $\mathrm{Der}_k(K)$. First we show that there exist $\delta_1, \dots, \delta_n \in \mathrm{Der}_k(K)$ and $x_1, \dots, x_n \in A$ such that $\delta_r(x_i) = 0$ for $1 \le i < r$, $\delta_r(x_r) = 1$ and $\delta_1, \dots, \delta_r, d_{r+1}, \dots, d_n$ is a basis of $\mathrm{Der}_k(K)$, for all $1 \le r \le n$. This we do by induction on r.

Since $d_1 \neq 0$, $d_1(A) \neq 0$. Therefore there exists $x_1 \in A$ such that $d_1(x_1) \neq 0$. Put $\delta_1 = (d_1(x_1))^{-1} d_1$. Suppose $r > 1$ and $\delta_1, \dots, \delta_{r-1}$ and $x_1, \dots, x_{r-1}$ have already been constructed. Then there exist (unique) elements $a_1, \dots, a_{r-1} \in K$ such that $\delta(x_i) = 0$ for $1 \le i < r$, where $\delta = d_r - \sum_{i=1}^{r-1} a_i \delta_i$. Since $\delta \neq 0$, there exists $x_r \in A$ such that $\delta(x_r) \neq 0$. Put $\delta_r = (\delta(x_r))^{-1}\delta$.

Since $\det(\delta_i(x_j)) \neq 0$, we may assume that $(\delta_i(x_j))$ is an $n \times n$ identity matrix.

Now we show that $\mathrm{Der}_k(A) \subset \sum_{i=1}^n A\delta_i$. Let $d \in \mathrm{Der}_k(A)$. Extend d to K. Then $d = \sum_{i=1}^n b_i \delta_i$, for some $b_i \in K$, $i = 1, \dots, n$. Then $b_i = d(x_i) \in A$ for all $i = 1, \dots, n$. Since A is Noetherian we are through. □

Example. Now we give an example of a non-regular local ring A containing a field k of characteristic zero such that $\mathrm{Der}_k(A)$ is finitely generated and free as an A-module (and A is differentially simple under $\mathrm{Der}_k(A)$). In fact, we take A to be the ring R constructed in [1, Example B]. For the sake of completeness, we give an alternative construction of A. Let X, Y be indeterminates over k and let d and δ be the k derivations of $k[X, Y]$ such that $d(X) = 0, d(Y) = 1$ and $\delta(X) = 1, \delta(Y) = 1 + Y$. Put $R = k[X, Y]_{(X,Y)}$. Then R is differentially simple under δ (see [6, Example 2.10]). For $a \in R$, let $n(a)$ denote the integer $n(a, XR + YR)$ as defined in Definition 3. Then by [3], $a \mapsto n(a)$ extends to a discrete valuation of $K = k(X, Y)$. Let B denote the valuation ring of this valuation. Then $R \subset B$. By [3], δ extends to B. Let

$$A = \{b \in B \mid d(b) \in B\}.$$

Then, A has the following properties:

(a) $R \subset A \subsetneq B$ and B is integral over A.

(b) $\delta(A) \subset A$ and A is differentially simple under δ.

(c) A is a Noetherian local ring of dimension 1.

(d) A is not regular and $\mathrm{Der}_k(A)$ is finitely generated and free.

Proof. (a) Since $d(R) \subset R$, $R \subset A$. Suppose $B = A$. Then d is a derivation of B. Since the maximal ideal of B is XB, and $d(X) = 0$, XB is d-differential. On the other hand, $Y \in XB$ and $d(Y) = 1 \notin XB$.

Now we show that B is integral over A. Let $b \in B$. By [4, Lemma 2.5], the natural map $R \to B/XB$ is surjective and hence $b = a + Xb'$ for some $a \in R$ and $b' \in B$. Therefore, we may assume that $b = Xb'$. Since $d(b') = c/X^n$ for some $c \in B$ and $n \geq 0$, hence $d(b^n) \in B$ i.e. $b^n \in A$.

(b) Since $[d, \delta] = d$ and $\delta(B) \subset B$, it follows that $\delta(A) \subset A$. Now, by [4, Lemma 2.5] A is differentially simple under δ.

(c) By (a), A is a local ring of dimension 1. By [4, Lemma 2.5], the maximal ideal of A is $XA + YA$. Hence A is Noetherian.

(d) Since $B \neq A$, A is not normal and hence not regular. Since the quotient field of A is $K = k(X, Y)$ and $\mathrm{Der}_k(K)$ is finitely generated as a K-vector space, by Lemma 9, $\mathrm{Der}_k(A)$ is finitely generated and hence by Corollary 7 free as an A-module. □

Remark 10. The above example is of interest in view of Zariski Lipman conjecture.

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